My research is in *Algebraic Number Theory*, more specifically in *Diophantine Analysis*, the study of Diophantine equations and inequalities. A Diophantine equation is an equation with integer (or rational) coefficients that is to be solved in integers (or rational numbers). A focus of study for hundreds of years, Diophantine Analysis remains a vibrant area of research. It has yielded a multitude of beautiful results and has wide ranging applications in other areas of mathematics, in cryptography, and in the natural sciences.

In my research, I prove that the equations in some specific families of Diophantine equations have no positive integer solutions. In general, I accomplish this by assuming a hypothetical solution exists and then applying various techniques to reach a contradiction. As a straightforward example, suppose that a, b, c are positive integers such that $2^a + 3^b = 4^c$. Since 2^a is even and 3^b is odd, the left side of the equation is odd while the right side is even, a contradiction. Therefore the equation $2^x + 3^y = 4^z$ has no solutions in positive integers. Though this type of argument is a common technique used throughout Diophantine Analysis, the problems I work on require more complicated methods.

Below, I present the three main theorems from my dissertation [16]. For each, I give some context and briefly describe the primary method that I use in the proof. In the next section, I state one of the results on bounding linear recurrences from my Master's research. Though I am not currently pursuing research topics in linear recurrences, it is topic with which I have some familiarity and may find useful in future collaborations. Finally, I describe my work in Diophantine Analysis since publishing my dissertation.

Dissertation Research in Diophantine Equations

Equations of the form $(x^k-1)(y^k-1) = (z^k-1)$ and $(x^k-1)(y^k-1) = (z^k-1)^2$ with $k \geq 4$, have no solutions [5]. In the theorem below, I prove that equations in a generalization of the second family have no solutions.

Theorem 1 *Let a, b, c, $k \in \mathbb{Z}^+$ with $k \geq 7$. The equation*

$$(a^2cx^k - 1)(b^2cy^k - 1) = (abcz^k - 1)^2$$

has no solutions in integers x, y, $z > 1$ with $a^2x^k \neq b^2y^k$.

I prove Theorem 1 using *Diophantine approximation*, the study of how closely an irrational number α can be approximated by a rational number β. There are numerous results providing lower bounds of $|\alpha - \beta|$ in terms of the denominator of β. One way to apply this method is to construct an α and β from a given hypothetical solution such that $|\alpha - \beta|$ is small. For example, in the proof of Theorem 1, I use $\alpha = \sqrt[k]{a^2c/(a^2cx^k-1)}$ and $\beta = y/z^2$. Applying known bounds on $|\alpha - \beta|$ for particular types of α can often result in a contradiction to most of the cases, usually leaving a finite number of possible values of α and β. For these, if the difference $|\alpha - \beta|$ is small enough then β must be a convergent of the continued fraction expansion of α. Then computations and standard properties of continued fractions may lead to a contradiction. (See [18] for details.)

Families of equations such as $X^2 + D = Y^N$, where D is a product of powers of a small number of primes have been studied for decades. More recently, there has been interest in the related family of equations, $NX^2 + 2^K = Y^N$, showing that under certain conditions there are no solutions [24,34].

Theorem 2 *Let $N > 1$ be an integer. Then the equation*

$$NX^2 + 2^L 3^M = Y^N$$

has no solutions with L, M, X, $Y \in \mathbb{Z}^+$ and $\gcd(NX, Y) = 1$.

I use *defective Lehmer pairs* in proving Theorem 2. A pair of algebraic integers is called a Lehmer pair if their quotient is not a root of unity and their product and square of their sum are nonzero coprime rational integers. The Lehmer pair is called t-defective, for $t \in \mathbb{Z}^+$, if the pair has a certain property depending on the divisors of a number constructed from the Lehmer pair. For almost all $t \in \mathbb{Z}^+$, the t-defective Lehmer pairs have been enumerated [4]. Comparing a t-defective Lehmer pair for some t, constructed from a hypothetical solution, to the list of known defective Lehmer pairs can lead to a contradiction [17].

The Tijdeman-Zagier conjecture states that $x^p + y^q = z^r$ has no solutions in positive integers when p, q, $r > 2$ and $\gcd(x, y, z) = 1$. Such equations are also called generalized Fermat equations. A survey of results can be found in [7]. For $N = 2$, 3, and 5 the family $X^{2N} + Y^2 = Z^5$ has no solutions [6,11,27]. I extend this work to all $N > 1$ when Y is a product of powers of 2, 5, and an arbitrary prime p.

Theorem 3 *Let p be an odd prime, $\alpha \geq 1$, and β, $\gamma \geq 0$ be integers. The equation*

$$X^{2N} + 2^{2\alpha} 5^{2\beta} p^{2\gamma} = Z^5$$

has no solutions with X, Z, $N \in \mathbb{Z}^+$, $N > 1$, and $\gcd(X, Z) = 1$.

The proof of Theorem 3, relies on the *modular approach*, a method that evolved from the proof of Fermat's Last Theorem. In particular, working with a fixed hypothetical solution, there exists an elliptic curve E (often called a *Frey curve*), whose coefficients depend only on the given solution. If E exists, then through rather deep results on Galois representation theory, E must arise from a newform, f of level N_n where N_n divides the conductor of E. The newform, f, captures valuable information about the solution that can be used to achieve a contradiction [19].

Master's Research in Linear Recurrences

The Fibonacci sequence is a very basic example of a sequence generated by a *second-order linear recurrence* or *difference equation.* In my Master's thesis, I studied both second-order recurrences and third-order difference equations. For each type of recurrence, I find explicit and often sharp upper bounds for the sequences, [8–10]. For example, below is the statement of a result giving an explicit bound on a second-order linear recurrence.

Theorem 4 *For $k > 1$, define the sequence $\{b_k\}$ with initial values $b_0 = 0$, $b_1 = -1$, and coefficients α_k, $\beta_k \in [0, A]$ by the recurrence*

$$b_k + \alpha_k b_{k-1} + \beta_k b_{k-2} = 0.$$

For a given $n \geq 76$, write $n - 1 = 15q + r$ with q, $r \in \mathbb{Z}$ such that $0 \leq r \leq 14$. If $\frac{2}{3} < A < \frac{3}{4}$, then $|b_n| \leq U_n$ where

$$U_n = \begin{cases} 2^{-4r+10\left[\frac{2r}{5}\right]+10} 3^{3q+3r-7\left[\frac{2r}{5}\right]-7} A^{9q+r-\left[\frac{2r}{5}\right]-1}, & \text{if } r \equiv 2 \pmod 5, \\ 2^{2r-5\left[\frac{2r}{5}\right]} 3^{3q-r+3\left[\frac{2r}{5}\right]} A^{9q+r-\left[\frac{2r}{5}\right]}, & \text{otherwise.} \end{cases}$$

Post-Dissertation Research

Since publishing my dissertation, I have spent time working on three distinct topics, Thue equations, Diophantine m-tuples, and arithmetic dynamics. My work on Thue equations has been the most productive. I will give a little bit of background of each topic and describe the results in what follows beginning with Thue equations.

In 1909, Axel Thue proved that if $F(X, Y)$ is an irreducible homogeneous polynomial of degree $n \geq 3$ with integer coefficients and m is a nonzero integer, then

$$F(X, Y) = m$$

has only finitely many solutions over the integers [31]. Individual such *Thue equations* of small degree can be solved over the integers using algorithms that are implemented in various computer algebra programs. (See [3,25,32,35] for details.)

Since 1990, starting with Thomas's result [30], several infinite families of Thue equations, parametrized by an integer t, have been completely solved over the integers (see Heuberger [20] for a survey). Of particular interest to me, is the extension of Thomas's work on parametrized families of simplest cubic, quartic, and sextic forms

$$\begin{aligned} F_t^3(X, Y) &= X^3 - (t-1)X^2Y - (t+2)X^2Y^2 - Y^3, \\ F_t^4(X, Y) &= X^4 - (t-1)X^3Y - 6X^2Y^2 + (t-1)XY^3 + Y^4, \text{ and} \\ F_t^6(X, Y) &= X^6 - 2(t-1)X^5Y - (5t+10)X^4Y^2 - 20X^3Y^3 + 5(t-1)X^2Y^4 \\ &+ (2t+4)XY^5 + Y^6 \end{aligned}$$

where t is imaginary quadratic number. The solutions of the *relative Thue equation* $F_t^3(X, Y) = \mu$, where X, $Y \in \mathbb{Z}_{\mathbb{Q}(t)}$, and μ is a root of unity in $\mathbb{Z}_{\mathbb{Q}(t)}$ can be found in the work of Heuberger, Pethő, and R.F. Tichy [21], and Heuberger [22].

More recently Gaál, Jadrijević, and Remete [15] solve the quartic and sextic families of Thue equations $F_t^k(X, Y) = \mu$ for $k = 4$, 6 where the parameter $t \in \mathbb{Z}$ rather than an imaginary quadratic integer but the solutions are elements of an imaginary quadratic number field.

Extensions of Thue equations over the integers where the equality in $F_t^k(X, Y) = \mu$ is replaced by a polynomial inequality have also been studied. For example, Lettl, Pethő, and Voutier [23] solve the families $F_t^k(X,Y) \leq p(t)$ over the integers, where $p(t)$ is an integral function of $t \in \mathbb{Z}$, and $k = 3, 4, 6$.

Working together with Dr. Faye, of Senegal and University of Witswatersrand and Dr. Wisniewski, of DeSales University [13], we are in the process of solving the two parametrized families of Thue inequalities $F_t^4(X,Y) \leq k(t)$ and $F_t^6(X,Y) \leq \ell(t)$. For each inequality, we are synthesizing the methods of Ziegler [36], Heuberger [22], Heuberger, Pethő, and Tichy [21], and Lettl, Pethő, and Voutier [23] and consider the cases in which $|t|$ is small and $|t|$ is large separately. In the case when $|t|$ is large, under the supposition that a nontrivial solution exists, we use the hypergeometric method together with a gap principle to obtain a contradiction. In the case when $|t|$ is small, Baker's method is combined with continued fractions in order to run a directed computer search for an expected finite number of nontrivial solutions.

A *Diophantine m-tuple* is a set of m integers with the property that the product of any two elements in the set plus one is a perfect square. Fermat found the first known Diophantine quadruple $\{1, 3, 8, 120\}$. An open problem in Diophantine m-tuples is the question of the existence of a Diophantine quintuple. It is conjectured that there are no Diophantine quintuples. One approach to this conjecture has been to begin with a Diophantine quadruple and show that there are no extensions to quintuples. The proofs often require solving simultaneous Pell equations.

Extending this idea to $\mathbb{Z}[\sqrt{d}]$ for some square-free integer d, we can define a *relative Diophantine m-tuple* to be a set of algebraic integers over $\mathbb{Q}(\sqrt{d})$. The cases $d = \pm 2$ have been studied by Abu Muriefah and Al-Rashed [1] and Franušić [14]. In this setting, Simon Rubinstein-Salzedo, Nitya Mani, and I consider various cases of positive and negative d and work to extend several triples to a quadruples This requires solving simultaneous Pell equations over $\mathbb{Z}[\sqrt{d}]$ as well as finding the integral points on quadratic twists of elliptic curves.

The study of *arithmetic dynamics* has been described as a blend of dynamical systems and Diophantine equations [28]. The orbit of x in a set S under a map $f : S \to S$ is the set of points obtained by repeated iteration of the map f to x. Bianca Thompson and I considered the orbit of a transcendental number α in $\mathbb{C}_p$ under the action of the Galois group of K_a over K, where K is a finite extension of the p-adic numbers $\mathbb{Q}_p$, K_a is an algebraic closure of K, and $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$.

Future Research

Huilin Zhu of Xiamen University, China and I have just begun collaboration on an extension of Theorem 2 along the same lines as his work with Le, Soydan, and Togbé [37]. We hope to employ innovative techniques along with well-known results such as those of Bilu, Hanrot, and Voutier [4].

New directions for my research include exploring different families of equations and new methods. For example, the family of equations $1^k + 2^k + \ldots + x^k = y^n$ has been studied for over a century. It has been solved for $1 \leq k < 170$ for n even [26], using local methods, the modular approach, linear forms in logarithms, elliptic curves, computer calculations, and previously known results. Some recent work has been done by Soydan [29] on $(x+1)^k + (x+2)^k + \ldots + (\ell x)^k = y^n$ for k, $\ell \geq 2$ and by Bérczes, Pink, Savaş, and Soydan [2] on the equation $(x+1)^k + (x+2)^k + \ldots + (x+d)^k = y^n$ for fixed positive integers k and d. Studying their work, I will examine variations of these families to see if they may yield some concrete results.

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